A Finite Element Formulation of Multi-layered Shells

for the Analysis of Laminated Composites

by

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Abstract

This paper presents a multi-layered/multi-director and shear-deformable finite element formulation of shells for the analysis of composite laminates. The displacement field is assumed continuous across the finite element layers through the composite thickness. The rotation field is, however, layer-wise continuous and is assumed discontinuous across these layers. This kinematic hypothesis results in independent shear deformation of the director associated with each individual layer and thus allows the warping of the composite cross-section. The resulting strain field is discontinuous across the different material sets, thereby creating the provision that the inter-laminar transverse stresses computed from the constitutive equations can be continuous. Numerical results are presented to show the performance of the method.

Contents

1. Introduction

In the last two decades, composites have found increasing application in many engineering structures. Recent advances in the technologies of manufacturing and materials have enhanced the current application of composite materials from being used as secondary structural elements to becoming primary load-carrying structural components. Due to the inherent inhomogeneity and anisotropy of the materials, analysis of these composite structures imposes new challenges on engineers [10, 11].

Plate and shell structures made of laminated composite materials have often been modeled as an equivalent single layer using classical laminate theory (C.L.T) [5, 8, 12, 13, 16] in which the thickness stress components are ignored. C.L.T. is a direct extension of classical plate theory in which the well known Kirchhoff-Love kinematic hypothesis is assumed enforced. This theory is adequate when the thickness (to side or radius ratio) is small. However, laminated plates and shells made of advanced filamentary composite materials are susceptible to thickness effects, because their effective transverse moduli are significantly small compared to the effective elastic modulus along the fiber direction. Furthermore, the classical theory of plates, which assumes that the normals to the mid-plane before deformation remain straight and normal to the plane after deformation, under predicts deflections and over predicts natural frequencies and buckling loads. These discrepancies arise due to the neglect of transverse shear strains. The range of applicability of the C.L.T. solution has been well established for laminated flat plates [10, 12, 20]. In order to overcome the deficiencies in C.L.T., refined laminate theories have been proposed [3, 10, 11, 18, 19, 20, 21, 24]. These are single layer theories in which the transverse shear stresses are taken into account. They provide improved global response estimates for deflections, vibration frequencies and buckling loads of moderately thick composites when compared to the classical laminate theory. A Mindlin type first-order transverse shear deformation theory (S.D.T.) was first developed by Whitney and Pagano [24] for multi-layered anisotropic plates, and by Dong and Tso [6] for multi-layered

anisotropic shells. Both approaches (C.L.T. and S.D.T.) consider all layers as one equivalent single anisotropic layer, thus they can not model the warping of cross-sections, that is, the in-plane distortion of the deformed normal due to transverse shear stresses. Furthermore, the assumption of a non-deformable normal results in incompatible shearing stresses between adjacent layers. The later approach also requires the introduction of an arbitrary shear correction factor which depends on the lamination parameters for obtaining accurate results. It is well established that such a theory is adequate to predict only the gross behavior of laminates.

The exact analyses performed by Pagano [12, 13, 16] on the composite flat plates have indicated that the in-plane distortion of the deformed normal depends not only on the laminate thickness, but also on the orientation and the degree of orthotropy of the individual layers. Therefore the hypothesis of non-deformable normals, while acceptable for isotropic plates and shells, is often quite unacceptable for multi-layered anisotropic plates and shells that have a large ratio of Young's modulus to shear modulus, even if they are relatively thin [4, 18, 19, 20]. Thus a *transverse shear deformation theory which also accounts for the warping of the deformed normal* is required for accurate prediction of the elastic behavior (deflections, thickness distribution of the in-plane displacements, natural frequencies, etc.) of multi-layered anisotropic plates and shells.

In view of these issues a variationally sound theory that accounts for the 3-D effects, allows thickness variation, and permits the warping of the deformed normal, is required for a refined analysis of thick and thin composites. In this paper we have endeavored to address these issues and develop a new finite element formulation for laminated composites. The assumed displacement field is continuous through the thickness, while the rotation field is layer-wise continuous (in 2-D) and can be discontinuous across the finite element layers. This displacement field fulfills *a priori* the geometric continuity conditions between contiguous layers. Furthermore, the assumed displacement field is capable of modeling the in-plane distortion of

the deformed normal, without increasing the order of the partial differential equations with respect to the first-order transverse shear deformation theory. In this formulation, at most, only first derivatives of displacement and rotation fields appear in the variational equations. The practical consequence of this fact is that only C^0 continuity of finite element functions is required which is readily satisfied by the Lagrange family of elements [7]. Because of the 3-D features in the formulation, it can accurately model the inter-laminar conditions and predict the 3-D edge effects. Finally, like the single-layer shear deformable theories, the proposed formulation provides flexibility in the specification of the boundary conditions [7].

A summary of the paper is as follows. Section 2 presents the assumptions inherent in the proposed model. The geometric and kinematic hypothesis of the multi-layered shells that accommodate, to an appropriate degree of approximation, the effects of transversal warping of the cross-section due to shear deformation as well as fiber compressibility are presented in Sections 3 and 4, respectively. Section 5 talks about the development of constitutive matrices for composite laminates. The underlying variational framework and the ensuing finite element formulation of the problem are presented in Section 6. Section 7 presents some numerical examples to demonstrate the range of applicability and accuracy of the proposed theory, and conclusions are drawn in Section 8.

2. Assumptions in the Layer-wise Shear Deformable Shell Theory

1. The domain Ω is of the following special form

$$\Omega = \left\{ (x, y, z) \in \mathcal{R}^3 | Z^{(l)} \in \left[\frac{-t}{2}, \frac{t}{2} \right]^{(l)}, T = \sum_l t^{(l)}, (x, y)^{(l)} \in A^{(l)} \subset \mathcal{R}^2 \right\} \quad (1)$$

where $A^{(l)}$ is the area of the reference surface for layer l, $t^{(l)}$ is the thickness of layer l and T is the total thickness of the composite shell.

2. The displacement field is assumed to take the following form

$$U_\alpha^{(l)}(x, y, z) = u_\alpha^{(l)}(x, y, z) + Z^{(l)} \theta_\alpha^{(l)}(x, y) \qquad \alpha = 1, 2 \quad (2)$$

where $u_\alpha^{(l)}(x, y, z)$ are the in-plane translations in layer l, $\theta_\alpha^{(l)}(x, y)$ are the out of plane director rotations for the reference surface associated with layer l, and $Z^{(l)}$ is a function that establishes the position of a point with respect to the reference surface in layer l. The second term on the right hand side of (2) can be viewed as an enhancement to the in-plane components of the displacement field in layer l with respect to the reference surface associated with the layer. Consequently this enhanced field becomes identically zero on the reference surfaces.

3. The displacement field in the thickness direction is assumed to be a function of z, i.e.,

$$U_3^{(l)}(x, y, z) = u_3^{(l)}(x, y, z) \tag{3}$$

thereby producing through the thickness strains which result in thickness variation in the shell. A consequence of the above relaxation is that $\sigma_{33} \neq 0$ i.e., we do not invoke the plane stress hypothesis.

Remarks:

1. The continuum requirement on the displacement field at the interface between l^{th} and $(l + 1)^{st}$ bounded layers necessitates the satisfaction of the contact conditions; $U_\alpha^{(l)} = U_\alpha^{(l+1)}$ (for $\alpha = 1, 2$), and $U_3^{(l)} = U_3^{(l+1)}$. These conditions are inherently satisfied via the definition of the assumed displacement field as defined in (2) and (3).

2. The inter-laminar stress continuity requires the satisfaction of the following conditions at the interface of lth and $(l+1)^{st}$ layers; $\tau_{\alpha 3}^{(l)} = \tau_{\alpha 3}^{(l+1)}$ (for $\alpha = 1, 2$), and $\tau_{33}^{(l)} = \tau_{33}^{(l+1)}$, where $\tau_{\alpha 3}^{(l)}$ denotes the inter-laminar stresses for layer l. It is important to note that the assumed displacement field (2) results in independent non-normal cross-sectional rotations in each finite element layer which is in accordance with the Mindlin kinematic assumption. A consequence is the independent shear deformation of the director in each layer, thus allowing the warping of the composite cross-section. Therefore the resulting strain field is discontinuous across different material sets and creates the provision of stress continuity across the interfaces. Consequently, the traction continuity equations are inherently satisfied.

3. Kinematic Description of Multi-layered Shells

Figure 1 shows a composite laminate with 'm' number of material layers. (For ease of presentation, in Fig. 1, $m = 3$). Consequently, the total thickness T is divided into 'm' elements through the thickness. Each layer of elements is associated with a reference surface which, for ease of discussion, is assumed to be coincident with the lower surface of that layer. It is important to note that the director rotation θ_α and the slope $\bar{u}_{3,\alpha}$ ($\bar{u}_3$ is the displacement field u_3 projected onto the reference surface for the layer) are not necessarily equal and thus transverse shear strains are accommodated. This is to be contrasted with the classical lamination theories in which $\theta_\alpha = \bar{u}_{3,\alpha}$. Within each layer the director rotates by $\theta_\alpha^{(l)}$, generating shear strain $\gamma_{\alpha3}^{(l)}$. Consequently, in the deformed configuration, node 2 (see Fig. 1) moves to location $2'$. Now, the director in the second layer rotates by an angle $\theta_\alpha^{(l+1)}$, generating shear strain $\gamma_{\alpha3}^{(l+1)}$. This kinematics is repeated in successive layers and, due to the continuity of the displacement field in the thickness direction, we obtain the new locations of nodal points (that constitute the layerwise directors) as $1'$, $2'$, $3'$ and $4'$. This new location of points produces a higher-order variation of the in-plane displacement field through the thickness. Accordingly, arbitrary polynomial expressions are not required to model the higher-order in-plane variation of displacement through the thickness of the composite.

3.1 Kinematics of an individual layer in the context of the finite element method

The displacement field of each layer l in the shell is defined by the following relations

$$u^{(l)}(\xi, \eta, \zeta) = \bar{u}^{(l)}(\xi, \eta) + U^{(l)}(\xi, \eta, \zeta) \tag{4}$$

$$\bar{u}^{(l)}(\xi, \eta) = \sum_{a=1}^{n_{en}} N_a(\xi, \eta)\, \bar{u}_a^{(l)} \tag{5}$$

$$U^{(l)}(\xi, \eta, \zeta) = \sum_{a=1}^{n_{en}} N_a(\xi, \eta)\, U_a^{(l)}(\zeta) \tag{6}$$

$$U_a^{(l)}(\zeta) = Z_a^{(l)}(\zeta)\, \widetilde{U}_a^{(l)} \quad \text{(no sum)} \tag{7}$$
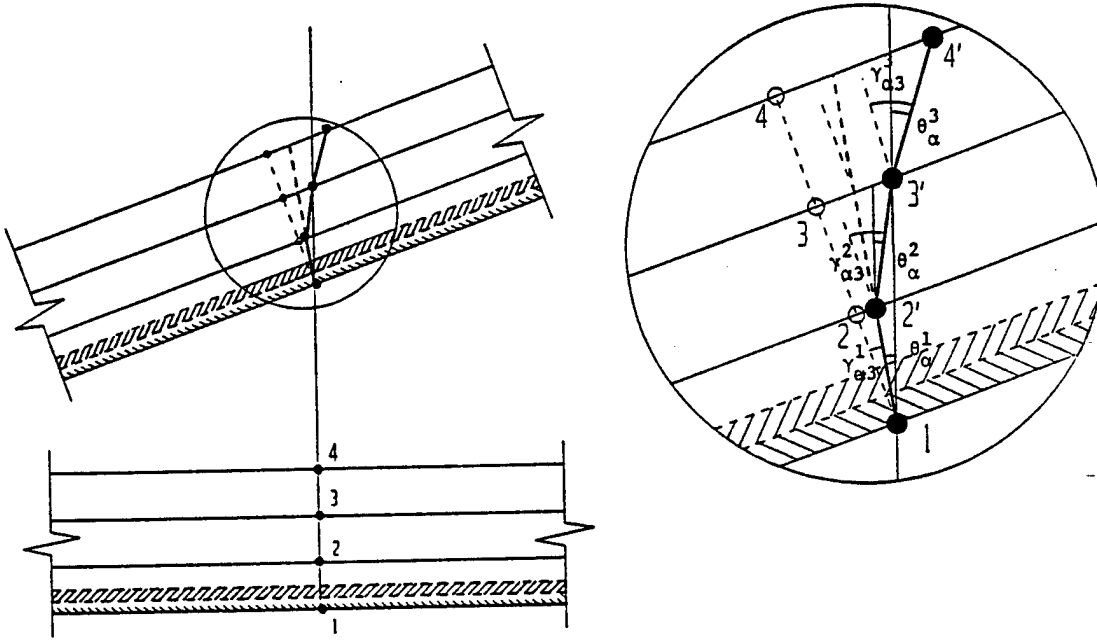
Fig. 1. Shell kinematics. Variation of transverse shear strains through the thickness.

$$\bar{u}_a^{(l)} = \frac{1}{2}\left(1 - \bar{\zeta}\right) u_a^{(l)-} + \frac{1}{2}\left(1 + \bar{\zeta}\right) u_a^{(l)+} \tag{8}$$

where $u_a^{(l)+}$ and $u_a^{(l)-}$ are the translations of the upper and lower surfaces of layer l, $\bar{u}^{(l)}$ is the displacement of a point on the reference surface of layer l, $U^{(l)}$ is the 'director displacement' for layer l, N_a represents two-dimensional shape function associated with node 'a', n_{en} are the number of element nodes, and $Z_a^{(l)}$ is the thickness function defined as

$$Z_a^{(l)}(\zeta) = \frac{1}{2}\left(1 + \zeta\right) Z_a^{(l)+} - \frac{1}{2}\left(1 - \zeta\right) Z_a^{(l)-} \tag{9}$$

where $Z_a^{(l)+} = \frac{1}{2}\left(1 - \bar{\zeta}\right) \|Z^{(l)}\|$ and $Z_a^{(l)-} = \frac{1}{2}\left(1 + \bar{\zeta}\right) \|Z^{(l)}\|$. Here $Z^{(l)} = x_a^{(l)+} - x_a^{(l)-}$ and $t^{(l)} = \|Z^{(l)}\|$ is the thickness of layer l. The parameter $\bar{\zeta} \in [-1, +1]$ defines the location of the reference surface. For example if $\bar{\zeta} = $ -1,0,+1 (respectively), then the reference surface is taken to be the bottom, middle, and top (respectively) of the shell layer l.

The nodal director displacement vector is constructed so that the director can rotate and

stretch as shown by an additive decomposition:

$$\tilde{U}_a^{(l)} = \hat{U}_a^{(l)} + \check{U}_a^{(l)} \tag{10}$$

The through thickness stretch component can be expressed as

$$\check{U}_a^{(l)} = \left(u_{a3}^{(l)^+} - u_{a3}^{(l)^-} \right) e_{a3}^{(l)f} \tag{11}$$

where $u_{a3}^{(l)^+}$ and $u_{a3}^{(l)^-}$ are the translations in the thickness direction of the upper and lower surfaces belonging to layer l. The out of plane rotation component of the director is constructed such that it may rotate, viz.,

$$\hat{U}_a^{(l)} = \theta_{a2}^{(l)} e_{a1}^{(l)f} - \theta_{a1}^{(l)} e_{a2}^{(l)f} \tag{12}$$

The quantities $\theta_{a1}^{(l)}$ and $\theta_{a2}^{(l)}$ represent the rotations of the director about the basis vectors $e_{a1}^{(l)f}$ and $e_{a2}^{(l)f}$ for shell layer l, respectively.

Remark: We have used the right-hand-rule sign convention for rotations.

4. Geometric Description of Multi-layered Shells

Figure 2 shows a typical configuration of a doubly curved composite shell. It can be made of numerous plies with variable material properties, ply orientations and thicknesses and can be stacked together in any arbitrary sequence. Figure 3 shows a schematic diagram of the geometry of a typical laminate of the composite shell (shown by the shaded region in Fig. 2) and can be defined by the following relations

$$x^{(l)}(\xi, \eta, \zeta) = \bar{x}^{(l)}(\xi, \eta) + X^{(l)}(\xi, \eta, \zeta) \tag{13}$$

$$\bar{x}^{(l)}(\xi, \eta) = \sum_{a=1}^{n_{en}} N_a(\xi, \eta) \, \bar{x}_a^{(l)} \tag{14}$$

$$X^{(l)}(\xi, \eta, \zeta) = \sum_{a=1}^{n_{en}} N_a(\xi, \eta) \, X_a^{(l)}(\zeta) \tag{15}$$

$$X_a^{(l)}(\zeta) = Z_a^{(l)}(\zeta) \, \hat{X}_a^{(l)} \quad \text{(no sum)} \tag{16}$$
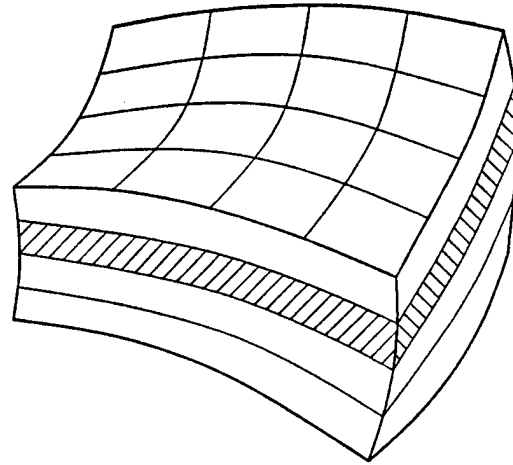
Fig. 2. Meso and macro structure of the composite.
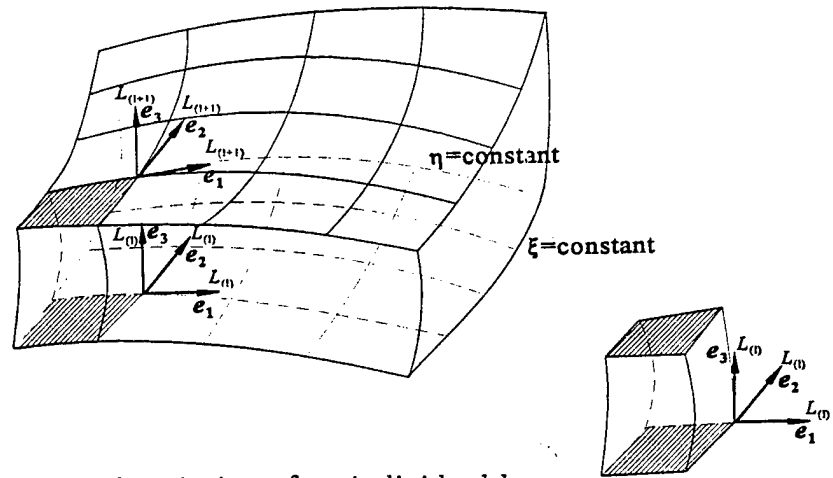

Fig. 3. Geometric description of an individual layer.

where $x^{(l)}$ is the position vector of a generic point in layer l, $\bar{x}^{(l)}$ is the position vector of a point on the reference surface for layer l, $X^{(l)}$ is the position vector of a generic point relative to $\bar{x}^{(l)}$ that defines the director through the point for layer l, (in computational shell literature, $X^{(l)}$ is referred to as fiber direction), $\hat{X}_a^{(l)} = Z^{(l)}/\|Z^{(l)}\|$ is a unit vector emanating from node 'a' in the director direction, and $Z_a^{(l)}$ is the thickness function associated with node 'a' as defined in (9). This function is defined by the location of the reference surface. $\bar{x}_a^{(l)}$ is the position vector of nodal point 'a' in layer l and is defined as

$$\bar{x}_a^{(l)} = \frac{1}{2}\left(1 - \bar{\zeta}\right) x_a^{(l)-} + \frac{1}{2}\left(1 + \bar{\zeta}\right) x_a^{(l)+} \tag{17}$$

5. Constitutive Relations

Most composites are made of a repeated sequence of laminates that have the same material properties but are oriented at $+\theta$ and $-\theta$ degrees with respect to the extension direction. Let $C'^{(l)}$ represent the linear constitutive matrix for an individual laminate with regard to its mutually perpendicular planes of elastic symmetry. This constitutive matrix for a laminate can be projected from its mutually perpendicular planes of elastic symmetry onto the global composite coordinate system via a transformation matrix $Q^{(l)}$ which is a function of the angle θ. The transformed constitutive matrix $C^{(l)}$ is obtained as

$$C^{(l)} = Q^{T(l)} C'^{(l)} Q^{(l)} \tag{18}$$

6. The Variational Framework

We start with the Hu-Washizu variational principle that uses the displacement, strain and stress fields as independent variables and thus provides the most general framework to formulate the problem and derive its weak form. We denote the strain field by ε and its variation by α. Similarly, we denote the stress field by σ and its variation by β. Since no boundary conditions are specified on the strain or stress fields, the solution space and the space of admissible variations coincide for both strains and stresses.

$$\mathcal{E} := \{\varepsilon \,|\, \varepsilon \in [\,\mathrm{L}_2(\Omega)\,]^{n_{sd}(n_{sd}+1)/2}\}$$

$$\mathcal{T} := \{\sigma \,|\, \sigma \in [\,\mathrm{L}_2(\Omega)\,]^{n_{sd}(n_{sd}+1)/2}\}$$

where n_{sd} denotes the number of spatial dimentions and for the present case $n_{sd} = 3$.

The Hu-Washizu formulation can be obtained by finding the stationary points of the functional

$$\Pi_{\mathrm{HW}}(\varepsilon, \sigma, u) := \frac{1}{2}\int_\Omega \varepsilon : C : \varepsilon\, d\Omega + \int_\Omega \sigma \cdot [\nabla^s u - \varepsilon]\, d\Omega - \Pi_{\mathrm{ext}}(u) \tag{19}$$

where the first term is the stored energy function, while the second term is a Lagrange multiplier σ enforcement of the strain-displacement relations. The assumed displacement

field (2)–(3) results in an enhanced strain field because layerwise rotations result in enhancing the shear strain components. We write the strain field as $\varepsilon = \nabla^s u + \tilde{\varepsilon}$, where $\nabla^s u$ is the symmetric compatible strain and $\tilde{\varepsilon}$ is the enhanced strain. The corresponding strain variation is $\alpha = \nabla^s \omega + \tilde{\alpha}$, where $\nabla^s \omega$ is the compatible and $\tilde{\alpha}$ is the enhanced strain variation. Since no boundary conditions are prescribed on the augmented strains as well, both spaces i.e., $\tilde{\varepsilon}$ and $\tilde{\alpha}$ conincide.

$$\tilde{\mathcal{E}} := \{\tilde{\varepsilon} \,|\, \tilde{\varepsilon} \in [\,L_2(\Omega)\,]^{n_{sd}(n_{sd}+1)/2} \}$$

We can now substitute this enhanced strain field in (19) to obtain an enhanced strain version of the Hu-Washizu formulation.

$$\Pi_{HW}(\tilde{\varepsilon}(\theta), \sigma, u) := \frac{1}{2} \int_\Omega (\nabla^s u + \tilde{\varepsilon}(\theta)) : C : (\nabla^s u + \tilde{\varepsilon}(\theta)) \, d\Omega - \int_\Omega \sigma \cdot \tilde{\varepsilon} \, d\Omega - \Pi_{ext}(u) \quad (20)$$

Following Simo and Hughes [22], we assume *a priori*, an orthogonality constraint on the stress field σ with respect to the enhanced strain field $\tilde{\varepsilon}(\theta)$. This causes the Lagrange multiplier, i.e., σ, to drop out from the formulation, thereby leading to a modified displacement model

$$\Pi(u, \theta) := \frac{1}{2} \int_\Omega (\nabla^s u + \tilde{\varepsilon}(\theta)) : C : (\nabla^s u + \tilde{\varepsilon}(\theta)) \, d\Omega - \Pi_{ext}(u, \theta) \quad (21)$$

The spaces relevant to the modified displacement formulation are

$$\mathcal{S} = \{(u, \theta) \in H^1(\Omega), (u, \theta) : \Omega \rightarrow \mathcal{R}^{n_{sd}}; u = \hat{u} \text{ on } \Gamma_u, \theta = \hat{\theta} \text{ on } \Gamma_\theta\} \quad (22)$$

$$\mathcal{V} = \{(w, \psi) \in H_0^1(\Omega), (w, \psi) : \Omega \rightarrow \mathcal{R}^{n_{sd}}\} \quad (23)$$

where $\mathcal{S}$ is the space of trial displacements and trial rotations, and $\mathcal{V}$ is the associated space of weighting functions, respectively. $H^1(\Omega)$ denotes the space of square-integrable functions along with their generalized derivatives defined over Ω, and $H_0^1(\Omega)$ is the subset of $H^1(\Omega)$ whose members satisfy zero essential boundary conditions.

For the case of small strains, assuming the composite laminate to occupy a region Ω in $\mathcal{R}^{n_{sd}}$, minimization of (21) leads to the following weak form: Given $f : \Omega \rightarrow \mathcal{R}^{n_{sd}}$,

$g : \Gamma_g \to \mathcal{R}^{n_{sd}-1}$, $h : \Gamma_h \to \mathcal{R}^{n_{sd}-1}$, find $\{u, \theta\} \in \mathcal{S}$ such that for all $\{\omega, \psi\} \in \mathcal{V}$

$$\Pi(u, \theta; \omega, \psi) = \int_\Omega (\nabla^s \omega + \tilde{\alpha}(\theta)) : C : (\nabla^s u + \tilde{\varepsilon}(\theta)) \, d\Omega - \int_\Omega \omega \cdot f \, d\Omega - \sum_{i=1}^{n_{sd}} \int_{\Gamma_h} \omega \cdot h \, d\Gamma \tag{24}$$

where $\nabla^s(\cdot)$ is the symmetric gradient operator, f denotes the body force vector defined over Ω, h represents the prescribed tractions (normal and shear) over boundary Γ_h, and g are the prescribed displacements and rotations defined over boundary Γ_g. Here $\Gamma_g = \Gamma_u \cup \Gamma_\theta$ with Γ_u being the boundary with prescribed displacements and Γ_θ being the boundary with prescribed rotations.

Now consider a shell consisting of 'm' generally directional but perfectly bonded layers, and reference surface of each layer $A^{(l)}$ discretized into 'n_{umel}' finite elements. Let $\Omega^{e(l)} = A^{e(l)} \times [-t/2, t/2]^{(l)}$, $\Omega^{(l)} = \bigcup_{e=1}^{n_{umel}} \Omega^{e(l)}$, and $\Omega = \sum_{l=1}^m \Omega^{(l)}$, where $\Omega^{e(l)}$ is the element domain in layer l. We can write the principle of virtual work (24) for the layered but perfectly bonded anisotropic composite laminates as

$$\tilde{\Pi}(u, \theta; \omega, \psi) = \sum_{l=1}^m \left\{ \int_{\Omega^{(l)}} (\nabla^s \omega + \tilde{\alpha}(\theta))^{(l)} : C^{(l)} : (\nabla^s u + \tilde{\varepsilon}(\theta))^{(l)} \, d\Omega \right.$$
$$\left. - \int_{\Omega^{(l)}} \omega^{(l)} \cdot f^{(l)} d\Omega - \sum_{i=1}^{n_{sd}} \int_{\Gamma_h^{(l)}} \omega^{(l)} \cdot h^{(l)} d\Gamma \right. \tag{25}$$

Let $\gamma = \nabla \bar{u}_3 - \theta$ be the shear strain vector, where $\bar{u}_3$ is the displacement field projected onto the reference surface for layer l. Similarly, let $\kappa = \nabla^s \theta$ be the curvature, which is also defined layer wise. We can write the energy functional (25) for the layered anisotropic laminates as

$$\tilde{\Pi}(u, \theta; \omega, \psi) = \sum_{l=1}^m \left\{ \int_{\Omega^{(l)}} \left(\nabla^s \omega^{(l)} : C_m^{(l)} : \nabla^s u^{(l)} + \bar{\gamma}^{(l)} : C_s^{(l)} : \gamma^{(l)} + \bar{\kappa}^{(l)} : C_b^{(l)} : \kappa^{(l)} \right) d\Omega \right.$$
$$\left. - \int_{\Omega^{(l)}} \omega^{(l)} \cdot f^{(l)} d\Omega - \sum_{i=1}^{n_{sd}} \int_{\Gamma_h^{(l)}} \omega^{(l)} \cdot h^{(l)} d\Gamma \right\} \tag{26}$$

where $\bar{\gamma}^{(l)}$ and $\bar{\kappa}^{(l)}$ are the virtual shear strains and curvatures, respectively. $\bar{\gamma}^{(l)} = \nabla \bar{\omega}_3^{(l)} - \psi^{(l)}$, and $\bar{\kappa}^{(l)} = \nabla^s \psi^{(l)}$, while $\bar{\omega}_3$ is the virtual displacement field projected onto the reference

surface for layer l. The assumed displacement field in (2) and (3) leads to a modified functional $\widehat{\Pi}$, defined as

$$\widehat{\Pi}(u, \theta; \omega, \psi) = \sum_{l=1}^{m} \left\{ \int_{\Omega^{(l)}} \bar{\varepsilon}^{(l)} : C_m^{(l)} : \varepsilon^{(l)} d\Omega + \int_{A^{(l)}} \bar{\gamma}^{(l)} : C_s^{(l)} : \gamma^{(l)} dA \right.$$

$$\left. + \int_{A^{(l)}} \bar{\kappa}^{(l)} : C_b^{(l)} : \kappa^{(l)} dA - \int_{\Omega^{(l)}} \omega^{(l)} \cdot f^{(l)} d\Omega - \sum_{i=1}^{n_{sd}} \int_{\Gamma_h^{(l)}} \omega^{(l)} \cdot h^{(l)} d\Gamma \right\} \quad (27)$$

where $\int_{\Omega^{(l)}} (\cdot) \, d\Omega$ is a volume integral and $\int_{A^{(l)}} (\cdot) \, dA$ represents an area integral.

We add the following boundary conditions to the energy functional (27) as

$$u_\alpha = \hat{u}_\alpha \qquad \text{on} \quad \Gamma_{u_\alpha} \qquad \alpha = 1, 2 \tag{28}$$

$$u_3 = \hat{u}_3 \qquad \text{on} \quad \Gamma_{u_3} \tag{29}$$

$$\theta_\alpha = \hat{\theta}_\alpha \qquad \text{on} \quad \Gamma_{\theta_\alpha} \qquad \alpha = 1, 2 \tag{30}$$

where $\Gamma_u = \Gamma_{u_\alpha} \cup \Gamma_{u_3}$ represents the boundary where translation boundary conditions are applied, and Γ_{θ_α} corresponds to the boundary with prescribed rotations. $\hat{u}_\alpha, \hat{u}_3,$ and $\hat{\theta}_\alpha$ represent the prescribed displacements and rotations, respectively. In addition, we prescribe the tractions as

$$\tau \cdot n = \hat{\tau} \qquad \text{on} \quad \Gamma_\tau \tag{31}$$

$$q \cdot n = \hat{q} \qquad \text{on} \quad \Gamma_q \tag{32}$$

$$m \cdot n = \hat{m} \qquad \text{on} \quad \Gamma_m \tag{33}$$

$\hat{\tau}$ and $\hat{q}$ represent the normal and shear tractions, respectively, while $\hat{m}$ represents the prescribed moments. This leads to a decomposition of Γ_h as $\Gamma_h = \Gamma_\tau \cup \Gamma_q \cup \Gamma_m$. $\Gamma_\tau \cup \Gamma_q$ represents the boundary where traction boundary conditions are applied, and Γ_m corresponds to the portion of boundary with applied moments.

The strain vectors for the proposed composite element are

$$\hat{\varepsilon} = \left\{ \begin{array}{c} \varepsilon_{11}^{(l)} \\ \varepsilon_{22}^{(l)} \\ 2\varepsilon_{12}^{(l)} \end{array} \right\}; \quad \gamma = \left\{ \begin{array}{c} \gamma_{13}^{(l)} \\ \gamma_{23}^{(l)} \end{array} \right\}; \quad \kappa = \left\{ \begin{array}{c} \kappa_{11}^{(l)} \\ \kappa_{22}^{(l)} \\ 2\kappa_{12}^{(l)} \end{array} \right\}; \quad \check{\varepsilon} = \left\{ \varepsilon_{33}^{(l)} \right\} \tag{34}$$

where $\hat{\varepsilon}, \gamma, \kappa$ and $\check{\varepsilon}$ represent the in-plane, shear, bending and through-thickness strains, respectively. We can combine the in-plane strains with through-thickness strains to yield

$$\varepsilon = \langle\, \varepsilon_{11}^{(l)} \;\; \varepsilon_{22}^{(l)} \;\; \varepsilon_{33}^{(l)} \;\; 2\varepsilon_{12}^{(l)} \,\rangle^T \tag{35}$$

The stresses corresponding to the strains are

$$\boldsymbol{\tau} = \left\{ \begin{array}{c} \tau_{11}^{(l)} \\ \tau_{22}^{(l)} \\ \tau_{33}^{(l)} \\ \tau_{12}^{(l)} \end{array} \right\}; \quad \boldsymbol{q} = \left\{ \begin{array}{c} q_1^{(l)} \\ q_2^{(l)} \end{array} \right\}; \quad \boldsymbol{m} = \left\{ \begin{array}{c} m_{11}^{(l)} \\ m_{22}^{(l)} \\ m_{12}^{(l)} \end{array} \right\} \tag{36}$$

where $\boldsymbol{\tau}$ represents the combined in-plane and the through-thickness stresses, and $\boldsymbol{q}$ and $\boldsymbol{m}$ represent the shear and bending stresses, respectively.

The element stiffness matrices and force vectors emanating from the weak form, that are obtained via Galerkin approximation and introduction of element shape functions can be written as

$$\boldsymbol{K} = \sum_{l=1}^{m} \left\{ \mathbf{A}_{e=1}^{n_{umel}} \boldsymbol{k}^{e(l)} \right\} \tag{37}$$

where $\mathbf{A}_{e=1}^{n_{umel}}$ is the finite element assembly operator. The element stiffness matrix can explicitly be written as

$$\boldsymbol{k}^{e(l)} = \int_{\Omega^{(l)}} \underbrace{\bar{\varepsilon}^{(l)} : C_m^{(l)} : \varepsilon^{(l)}}_{\text{modified membrane}} \, d\Omega + \int_{A^{(l)}} \underbrace{\bar{\gamma}^{(l)} : C_s^{(l)} : \gamma^{(l)}}_{\text{shear}} \, dA + \int_{A^{(l)}} \underbrace{\bar{\kappa}^{(l)} : C_b^{(l)} : \kappa^{(l)}}_{\text{bending}} \, dA \tag{38}$$

where $C_m^{(l)}, C_s^{(l)}$ and $C_b^{(l)}$ are the modified-membrane, shear and bending constitutive matrices for layer l, respectively. The right hand side force vector can be written as

$$\boldsymbol{F} = \sum_{l=1}^{m} \left\{ \mathbf{A}_{e=1}^{n_{umel}} \left(\int_{\Omega^{e(l)}} \omega^{(l)} \cdot f^{(l)} d\Omega + \sum_{i=1}^{n_{sd}} \int_{\Gamma_h^{e(l)}} \omega^{(l)} \cdot h^{(l)} d\Gamma \right) \right\} \tag{39}$$

where the first term is to the body force vector and the second term is the traction vector.

Remark: We have used selective reduced integration concepts for the evaluation of the element stiffness matrices.

7. Numerical Examples

7.1 Free edge boundary-value problem. [45,-45]s

The first numerical simulation is a prismatic symmetric laminate having traction-free edges at $Y = \pm a$ and surfaces $Z = \pm h$, and is subjected to strain increment only on its ends at $X = $ constant. Each layer is composed of uni-directional fiber-reinforced material such that the fiber direction is defined by its angle θ with respect to the X-axis. The elastic properties used in this simulation are taken from Pagano [12], p. 4. The laminate consists of four uni-directional fibrous composite layers, two with axis of elastic symmetry at +45 and two at −45 to the longitudinal laminate axis. (See Fig. 4.) 1% strain in opposite directions is applied at $X = \pm L$, while it is restrained to move in the axial, lateral and thickness directions at $X = 0$. The physical dimensions for the numerical simulation are $X = 60, Y = 20, Z = 2.5$, with 12 elements in X direction, 40 elements in the Y direction and 4 elements through the thickness. For each finite element layer through the thickness, the reference surface is assumed to be coincident with the bottom surface of the layer.

Figure 5 shows the axial displacement distribution at the top free surface of the section cut at $X = 0$, and the results are compared with Moires *et al.* and Pagano *et al.* [12]. Figure 6 presents the inter-laminar shear stress at material interface and their corresponding values from Reddy [20]. Figure 7 shows the major stress components σ_{11}, σ_{12}, σ_{13}, and their corresponding values from the numerical simulations of Pipes *et al.* [16]. In order to complete the discussion, we have also presented the minor stress components in Fig. 8.
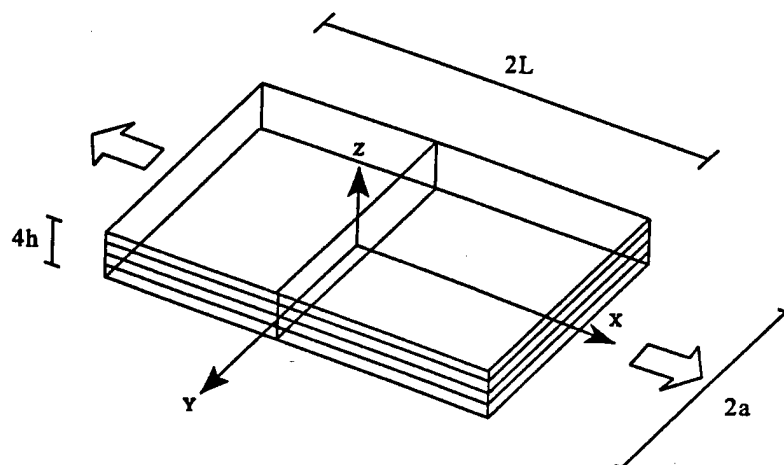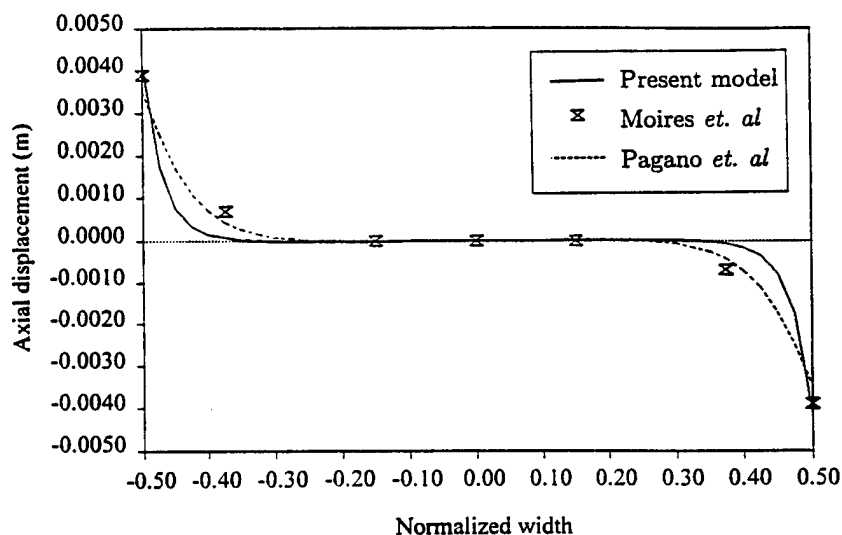
Figure 4. Laminate geometry and the coordinate system.

Figure 5. Axial displacement distribution at the top free surface.

7.2　Free edge boundary-value problem with a cylindrical hole. [45,-45]s

The composite laminate considered in this simulation has the same physical dimensions, material properties and loading conditions as in the previous case. However the present laminate has a unit diameter cylindrical hole at (0,0,0) with its axis coincident with the Z axis (see Fig. 9). The ratio of the diameter of the hole to the width of the laminate is 0.1. Traction-free boundary conditions are applied on the cylindrical surface of the hole. The
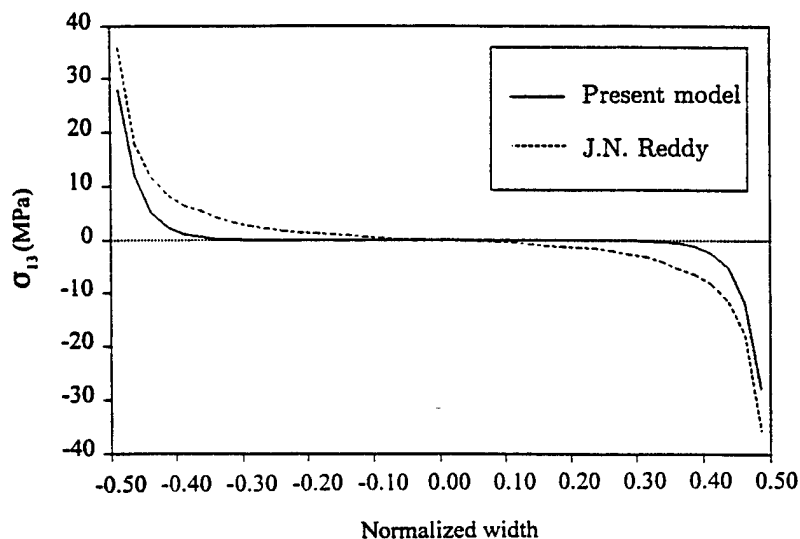
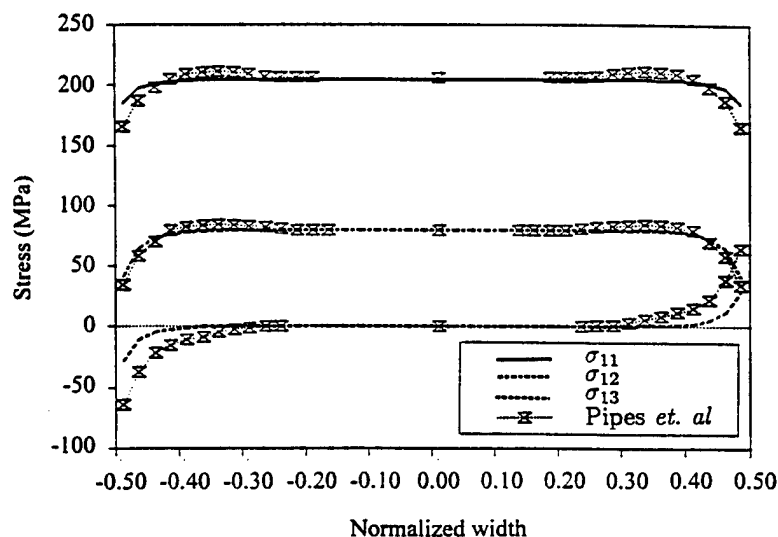Figure 6. Interlaminar shear stress at material interface.

Figure 7. Major stress components in the top +45 laminate.

laminate is constrained to move in the axial, lateral and transverse directions by appropriately constraining the nodes along the symmetry lines at $X = 0$.

Figure 10 presents the major stress components σ_{11}, σ_{12}, σ_{13} at a section cut at $X = 0$ in the top layer with +45 degrees ply orientation. These results have been compared with complete 3D anisotropic calculations done with a mesh which is twice as refined as the present mesh. In the region away from the free edge boundary and the traction free hole, the ratio of
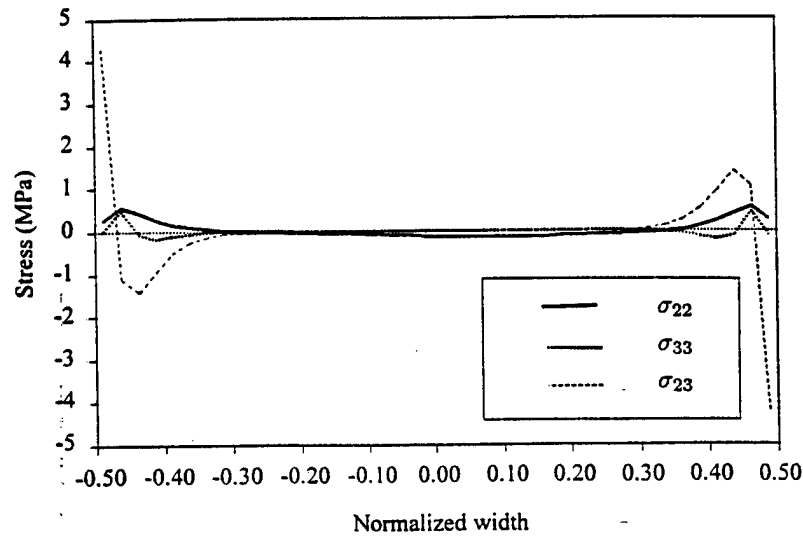
Figure 8. Minor stress components in the top +45 laminate.

σ_{11} and σ_{12} agrees closely with that of the preceding numerical simulation. Figure 11 shows the minor stress components which also show a good agreement with the 3D anisotropic simulation.
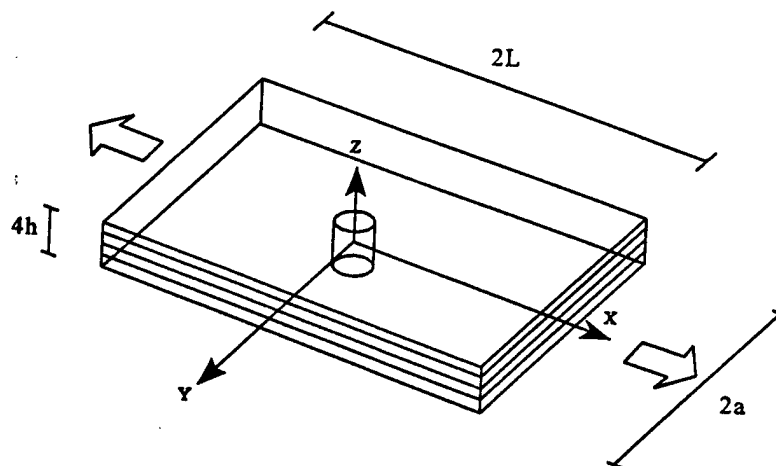

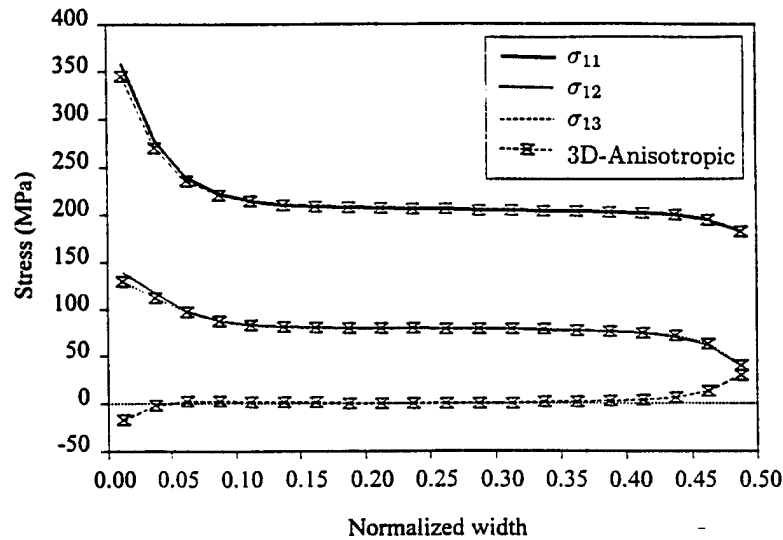Figure 9. Laminate geometry with a cylindrical hole.

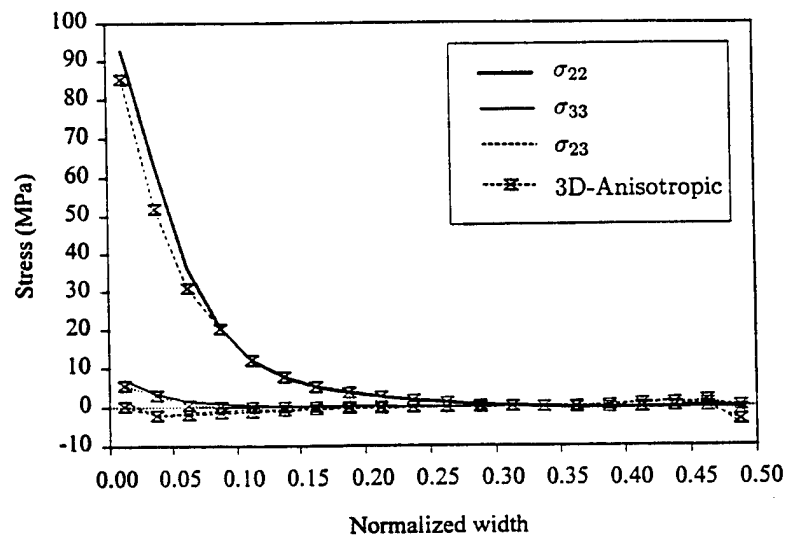Figure 10. Major stress components in the top +45 laminate.

Figure 11. Minor stress components in the top +45 laminate.

8. Conclusions

In this paper we have presented a finite element formulation which is suitable for the analysis of multi-layered/multi-director and shear-deformable composite laminates. The geometric description employed for the composite shells finds its roots in the so-called degenerated shell approach. A set of kinematic hypothesis is introduced to accommodate the effects of transverse warping of the cross section due to shear deformation along with fiber com-

pressibility that results from transverse normal stresses. The kinematics are individually and independently represented for each layer and are coupled via the conditions of displacement continuity between contiguous layers. The rotation field is however layer-wise continuous and is assumed discontinuous across the layers. Transversal warping of the composite cross-section is described by individual layer directors which simultaneously rotate and stretch. This results in discontinuous strain fields across the different material sets, thereby creating the provision of stress continuity across the material interfaces. Since the formulation resolves all strains, all stresses are computed through the three-dimensional constitutive equations and the usual 'zero normal stress' shell hypothesis is not employed. The variational equation contains only the first derivatives of displacement and rotation fields that require just the C^0 continuity of finite element functions [7]. In displacement formulation of plates and shells, special attention needs to be given to transverse shear and membrane terms to prevent the occurrence of *mesh locking*. We have employed the selective/reduced integration technique to avoid this problem. Numerical results are presented that demonstrate the good performance of the proposed formulation.

Acknowledgements

References

[1] J. Argyris and L. Tenek, "An efficient and locking-free flat anisotropic plate and shell triangular element", *Comput. Methods in Appl. Mech. Engrg.*, **118**, 63–119, 1994.

[2] W.S. Burton and A.K. Noor, "Assessment of computational models for sandwich panels and shells", *Comput. Methods in Appl. Mech. Engrg.*, **124**, 125–151 1995.

[3] C. Byun and R.K. Kapania, "Prediction of interlaminar stresses in laminated plates using global orthogonal interpolation polynomials", *AIAA Journal*, **30**, No 11, November 1992.

[4] Fu-Kuo Chang, J.L. Perez and K.Y. Chang, "Analysis of thick laminated composites", *Journal of Composite Materials*, **24**, 801–822, August 1990.

[5] R.M. Christensen, *Mechanics of Composite Materials*, John Wiley & Sons, NY 1979.

[6] S.B. Dong and F.K.W Tso, "On a laminate orthotropic shell theory including transverse shear deformation", *Journal of Applied Mechanics*, **39**, 1972.

[7] T.J.R. Hughes, *The Finite Element Method: Linear Static and Dynamic Finite Element Analysis*, Prentice-Hall, Englewoods Cliffs, NJ, 1987.

[8] R.M. Jones, *Mechanics of Composite Materials*, McGraw-Hill Book Co., New York, 1975.

[9] S.S. Lekhnitskii, *Theory of Elasticity of an Anisotropic Body*, Mir Publishers, Moscow, 1981.

[10] A.K. Noor and W.S. Burton, "Assessment of shear deformation theories for multi-layered composite plates", *Applied Mechanics Review*, **42**, No. 1, 1–13, 1989.

[11] A.K. Noor and W.S. Burton, "Assessment of computational models for multi-layered composite shells", *Applied Mechanics Review*, **43**, No. 4, 67–97 1990.

[12] N.J. Pagano, *Interlaminar response of composite materials*, Composite Materials Series, Vol. 5., Elsevier, New York, NY, 1989.

[13] N.J. Pagano and R.B. Pipes, "Some observations on the interlaminar strength of composite laminates", *Int. J. Mech. Sci.*, **15**, 679, 1973.

[14] P.M. Pinsky and R.V. Jasti, "A mixed finite element for laminated composite plates based on the use of bubble functions", *Engineering Computations*, **6**, 316-330, 1989.

[15] P.M. Pinsky and K.O. Kim, "A multi-director formulation for elastic-viscoelastic layered shells", *Internat. J. Numer. Methods Engrg.*, **23**, 2213-2244, 1986.

[16] R.B. Pipes and N.J. Pagano, "Interlaminar stresses in composite laminates under uniform axial extension", *J. Comp. Materials*, **4**, 538–548, 1970.

[17] L. Vu-Quoc and H. Deng, "Galerkin projection for geometrically exact sandwich beams allowing for ply drop-off", *J. of Appl. Mech.*, **62**, No.2, 479–488, 1995.

[18] J.N. Reddy, "A simple higher-order theory for laminated composite plates", *Journal of Applied Mechanics*, **51**, 745–752, December 1984.

[19] J.N. Reddy, "A refined non-linear theory of plates with transverse shear deformation", *Int. J. Solids Struct.*, **20**, 881–896, October 1984.

[20] J.N. Reddy, "An evaluation of equivalent-single-layer and layer-wise theories of composite laminates", *Composite Structures*, **25**, 21–35, 1993.

[21] D.H. Robbins, J.N. Reddy and A.V.K. Murty, "On the modeling of delamination in thick composites" in *Enhanced Analysis Techniques for Composite Materials*, ed. L. Schwer, J.N. Reddy and A. Mal, *The Winter Annual Meeting of the American Society of Mechanical Engineers*, Atlanta, Georgia, December 1-6, 1991.

[22] J.C. Simo and T.J.R. Hughes, "On the variational foundations of assumed strain methods", *J. of Appl. Mech.*, **53**, 51–54, 1986.

[23] J.M. Whitney, *Structural Analysis of Laminated Anisotropic Plates*, Technomic Publishing Company, Inc. PA, 1987.

[24] J.M. Whitney and N.J. Pagano, "Shear deformation in heterogeneous anisotropic plates," *J. App. Mech.*, **37**, 1031, 1970.